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Closure fuzzy ideals of MS-algebras

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Abstract. In this paper, we introduce concepts of dominator fuzzy ideals and closure fuzzy ideals in MS-algebras and study some properties of these fuzzy ideals. It is proved that the lattice of all closure fuzzy ideals is isomorphic to the fuzzy ideal lattice of the lattice of all principal dominator ideals. A set of equivalent conditions is obtained to characterize closure fuzzy ideals of MS-algebras. Finally some properties of closure fuzzy ideals are studied with respect to homomorphisms.

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1. Introduction

An Ockham algebra is a bounded distributive lattice with a dual endomorphism. The class of all Ockham algebras contains the well-known classes for examples Boolean algebras, de Morgan algebras, Kleene algebras and Stone algebras [5]. Blyth and Varlet [6] defined a subclass of Ockham algebras so called MS-algebras which generalize both de Morgan algebras and Stone algebras. These algebras belong to the class of Ockham algebras introduced by Berman [4]. The class of all MS-algebras forms an equational class. Blyth and Varlet [7] characterized the subvarieties of MS-algebras. Recently, Rao [13] introduced the notion of e-filters of MS-algebras, Rao [12] introduced the concepts of boosters and β-filters of MS-algebras and also Badawy and Rao [3] introduced the clousure ideals of MS-algebras.
On the other hand, fuzzy set theory introduced by Zadeh [17] is generalization of classical set theory. Next, Rosenfeld [11] applied it to group theory and developed the theory of fuzzy subgroups. Also, many authors have worked on fuzzy lattice theory. They introduced the concepts of fuzzy sublattice, fuzzy ideal, fuzzy prime ideal, in a lattice and gave some interesting results (see [1, 2, 9, 10, 14, 16]. In this paper, we introduce dominator fuzzy subsets and dominator fuzzy ideals and study some

basic properties of dominator fuzzy subsets and dominator fuzzy ideals. Moreover the concept of closure fuzzy ideals is introduced in MS-algebras. Some properties of closure fuzzy ideals of an MS-algebra are observed. It is proved that the class $FI_C(L)$ of all closure fuzzy ideals of an MS-algebra L is a bounded distributive lattice with set inclusion. Also it is proved that $FI_C(L)$ is isomorphic to the ideal lattice of $FI(M_{\circ\circ}(L))$. Finally, some properties of closure fuzzy ideals are studied with respect to homomorphisms. The concept of dominator fuzzy ideal preserving homomorphism from an MS-algebra L into another MS-algebra M is introduced as a homomorphism f satisfying the condition $f(\mu_{\circ\circ}) = \{f(\mu)\}_{\circ\circ}$, for any fuzzy ideal μ of L. It is proved that the images and the inverse images, under this homomorphism, of closure fuzzy ideals are again closure fuzzy ideals.

2. Preliminaries

In this section, we recall some definitions and results which will be used in this paper. For in details in ordinary crisp theory of closure ideals of MS-algebras, we refer to [3].

Definition 2.1 ([6])**.** An MS-algebra is an algebra $(L, \vee, \wedge, ^\circ, 0, 1)$ of type $(2, 2, 1, 0, 0)$, such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $a \to a^\circ$ is a unary operation satisfies: $a \leq a^{\circ\circ}$, $(a \wedge b)^\circ = a^\circ \vee b^\circ$ and $1^\circ = 0$.

A Stone algebra $S = (S, \vee, \wedge, ^*, 0, 1)$ is also a bounded distributive lattice, endowed with a unary operation $x \to x^*$ satisfying $(x \wedge y)^* = x^* \vee y^*, x \wedge x^* = 0$ and $0^* = 1$.

A de Morgan algebra is an algebra $(L, \vee, \wedge, ^-, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $a \to \overline{a}$ is a unary operation satisfies: $\overline{\overline{a}} = a$, $\overline{(a \wedge b)} = \overline{a} \vee \overline{b}$ and $\overline{1} = 0$.

Lemma 2.2 ([6])**.** *For any two elements a, b of an MS-algebra, we have the following:*

(1) $0^\circ = 1$,
(2) $a \leq b \Rightarrow b^\circ \leq a^\circ$,
(3) $a^{\circ\circ\circ} = a^\circ$,
(4) $(a \vee b)^\circ = a^\circ \wedge b^\circ$,
(5) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$,
(6) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$.

Definition 2.3 ([3])**.** For any nonempty subset A of an MS-algebra L, define the dominator of A as follows: $A_{\circ\circ} = \{x \in L : x \leq a^{\circ\circ} \text{ } for \text{ } some \text{ } a \in A\}$.

Note that $\{0\}_{\circ\circ} = \{0\}$ and $L_{\circ\circ} = L$.

Definition 2.4 ([3])**.** An ideal I of an MS-algebra L is called a dominator ideal, if $I = I_{\circ\circ}$.

The set of all dominator ideals of L denoted by $I_{\circ\circ}(L)$ and $I_{\circ\circ}(L)$ form a bounded distributive lattice with set inclusion.

Definition 2.5 ([8])**.** For any elements a in a lattice L, the principal ideal of L generated by a, $(a]$, defined by $(a] = \{x \in L : x \leq a\}$.

Definition 2.6 ([3]). For any element a of an MS-algebra L, the dominator $\{a\}_{\circ\circ}$ is called a principal dominator ideal.

Note that $\{0\}_{\circ\circ} = \{0\}$ and $\{1\}_{\circ\circ} = L$ and for any $a \in L$, $\{a\}_{\circ\circ} = (a]_{\circ\circ} = (a^{\circ\circ}]$. The set of all principal dominator ideals of L denoted by by $M_{\circ\circ}(L)$, *i.e.*, $M_{\circ\circ}(L) = \{(a]_{\circ\circ} : a \in L\}$. $M_{\circ\circ}(L)$ is a bounded sublattice of $I_{\circ\circ}(L)$ and it is a de Morgan algebra.

Definition 2.7 ([3]). Let L be an MS-algebra. For any ideal I of L, define an operator $\sigma : I(L) \to I(M_{\circ\circ}(L))$ as follows: $\sigma(I) = \{(x]_{\circ\circ} : x \in I\}$.

Definition 2.8 ([3]). Let L be an MS-algebra. For any ideal $\overline{I}$ of $M_{\circ\circ}(L)$, define an operator $\overleftarrow{\sigma} : I(M_{\circ\circ}(L)) \to I(L)$ as follows: $\overleftarrow{\sigma}(\overline{I}) = \{x \in L : (x]_{\circ\circ} \in \overline{I}\}$.

We recall that for any nonempty set L, the characteristic function of L,

$$\chi_L(x) = \begin{cases} 1 & if\ x \in L, \\ 0 & if \quad x \notin L. \end{cases}$$

Definition 2.9 ([3]). An ideal I of an MS-algebra L is a closure ideal, if $\overleftarrow{\sigma}\sigma(I) = I$.

Definition 2.10 ([2]). Let μ be a fuzzy subset of $(L, \wedge, \vee, 0, 1)$. For any $\alpha \in [0, 1]$, we shall denote the level subset $\mu^{-1}([\alpha, 1])$ by simply μ_α, i.e.,

$$\mu_\alpha = \{x \in L : \alpha \leq \mu(x)\}.$$

A fuzzy subset μ of L is proper, if it is a non constant function. A fuzzy subset μ such that $\mu(x) = 0$, for all $x \in L$ is an improper fuzzy subset.

Definition 2.11 ([11]). Let μ and θ be fuzzy subsets of a set L. Define the fuzzy subsets $\mu \cup \theta$ and $\mu \cap \theta$ of L as follows: for each $x \in L$,

$$(\mu \cup \theta)(x) = \mu(x) \vee \theta(x) \text{ and } (\mu \cap \theta)(x) = \mu(x) \wedge \theta(x).$$

Then $\mu \cup \theta$ and $\mu \cap \theta$ are called the union and intersection of μ and θ respectively.

We define the binary operations " $\vee$ " and " $\wedge$ " on all fuzzy subsets of L as:

$$(\mu \vee \theta)(x) = \sup\{\mu(a) \wedge \theta(b) : a, b \in L, a \vee b = x\}$$

and

$$(\mu \wedge \theta)(x) = \sup\{\mu(a) \wedge \theta(b) : a, b \in L, a \wedge b = x\}.$$

If μ and θ are fuzzy ideals of L, then $\mu \wedge \theta = \mu \cap \theta$ and $\mu \vee \theta$ is a fuzzy ideal generated by $\mu \cup \theta$.

Theorem 2.12 ([14]). *Let μ be a fuzzy subset of L. Then μ is a fuzzy ideal of L if and only if any one of the following conditions are satisfied:*

(1) $\mu(0) = 1$ *and* $\mu(x \vee y) = \mu(x) \wedge \mu(y),\ for\ all\ x, y \in L,$
(2) $\mu(0) = 1$ *and* $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$ *and* $\mu(x \wedge y) \geq \mu(x) \vee \mu(y),\ for\ all\ x, y \in L.$

Proposition 2.13 ([2]). *Let μ is a fuzzy ideal of L. If $x \leq y$, then $\mu(x) \geq \mu(y),\ for\ all\ x, y \in L.$*

Theorem 2.14 ([14]). *Let μ be a fuzzy subset of L. Then μ is a fuzzy ideal of L if and only if for any $\alpha \in [0, 1]$ such that $\mu_\alpha \neq \varnothing, \mu_\alpha$ is an ideal of L.*

Definition 2.15 ([2]). A proper fuzzy ideal μ of L is called prime fuzzy ideal, if for any two fuzzy ideals η, ν of L, $\eta \cap \nu \subseteq \mu$ implies $\eta \subseteq \mu$ or $\nu \subseteq \mu$.

Theorem 2.16 ([15]). *For any prime ideal p of L and $\alpha \in [0, 1)$ if and only if there exists a prime fuzzy ideal P^1_α of L is given by: for each $x \in L$,*

$$P^1_\alpha(x) = \begin{cases} 1 & if\ x \in P, \\ \alpha & if\ \ x \notin P. \end{cases}$$

Theorem 2.17 ([16]). *Let $f : L \to L^{'}$ be an onto homomorphism. Then $f(\mu)$ is a fuzzy ideal of $L^{'}$ if μ is a fuzzy ideal of L.*

Theorem 2.18 ([1]). *Let $f : L \to L^{'}$ be homomorphism. Then $f^{-1}(\mu)$ is a fuzzy ideal of L, if μ is a fuzzy ideal of $L^{'}$.*

Throughout in the next sections L stands for an MS-algebra unless otherwise mentioned.

3. Dominator fuzzy sets, dominator fuzzy ideals of MS-algebras

In this section, the concepts of dominator fuzzy subsets and dominator fuzzy ideals are introduced in MS-algebras. Some properties of these fuzzy ideals are investigated.

Definition 3.1. For any non-empty fuzzy subset μ of an MS-algebra L, define the dominator of μ as follows: $\mu_{\circ\circ}(x) = \sup\{\mu(a) : x \leq a^{\circ\circ}, a \in L\}$, for all $x \in L$

Obviously, $\{\chi_{\{0\}}\}_{\circ\circ} = \chi_{\{0\}}$ and $\{\chi_L\}_{\circ\circ} = \chi_L$.

Lemma 3.2. *For any two fuzzy subsets μ, θ of MS-algebra L, we have the following:*

(1) $\mu \subseteq \mu_{\circ\circ}$,
(2) $\mu \subseteq \theta$ *implies* $\mu_{\circ\circ} \subseteq \theta_{\circ\circ}$,
(3) $\{\mu_{\circ\circ}\}_{\circ\circ} = \mu_{\circ\circ}$.

Proof. (1) $\mu_{\circ\circ}(x) = \sup\{\mu(a) : x \leq a^{\circ\circ}, a \in L\} \geq \mu(x)$, since $x \leq x^{\circ\circ}$. Then $\mu \subseteq \mu_{\circ\circ}$.

(2) Suppose $\mu \subseteq \theta$. Then for all $x \in L$,

$$\mu_{\circ\circ}(x) = \sup\{\mu(a) : x \leq a^{\circ\circ}, a \in L\} \leq \sup\{\theta(a) : x \leq a^{\circ\circ}, a \in L\} = \theta_{\circ\circ}(x).$$

Thus $\mu_{\circ\circ} \subseteq \theta_{\circ\circ}$.

(3) Since $a^{\circ\circ} \leq y^{\circ\circ\circ\circ} = y^{\circ\circ}$,

$$\begin{aligned}\{\mu_{\circ\circ}\}_{\circ\circ}(x) &= \sup\{\mu_{\circ\circ}(a) : x \leq a^{\circ\circ}\} \\ &= \sup\{\sup\{\mu(y) : a \leq y^{\circ\circ}\} : x \leq a^{\circ\circ}\} \\ &= \sup\{\mu(y) : x \leq y^{\circ\circ}\} = \mu_{\circ\circ}(x).\end{aligned}$$

Then $\{\mu_{\circ\circ}\}_{\circ\circ} = \mu_{\circ\circ}$. □

Lemma 3.3. *For any two fuzzy ideals μ and θ of MS-algebra L, the following hold:*

(1) $\mu_{\circ\circ}$ *is a fuzzy ideal of* L,
(2) $(\mu \cap \theta)_{\circ\circ} = \mu_{\circ\circ} \cap \theta_{\circ\circ}$,
(3) $(\mu \vee \theta)_{\circ\circ} = \mu_{\circ\circ} \vee \theta_{\circ\circ}$.

Proof. (1) Clearly, $\mu_{\circ\circ}(0) = 1$. Let $x, y \in L$. Then

$$\begin{aligned}\mu_{\circ\circ}(x) \wedge \mu_{\circ\circ}(y) &= \sup\{\mu(a) : x \leq a^{\circ\circ}, a \in L\} \wedge \sup\{\mu(b) : y \leq b^{\circ\circ}, b \in L\}\\ &= \sup\{\mu(a) \wedge \mu(b) : x \leq a^{\circ\circ}, y \leq b^{\circ\circ}\}\\ &\leq \sup\{\mu(a \vee b) : x \vee y \leq a^{\circ\circ} \vee b^{\circ\circ} = (a \vee b)^{\circ\circ}, a, b \in L\}\\ &= \mu_{\circ\circ}(x \vee y)\end{aligned}$$

and

$$\begin{aligned}\mu_{\circ\circ}(x) \vee \mu_{\circ\circ}(y) &= \sup\{\mu(a) : x \leq a^{\circ\circ}, a \in L\} \vee \sup\{\mu(b) : y \leq b^{\circ\circ}, b \in L\}\\ &= \sup\{\mu(a) \vee \mu(b) : x \leq a^{\circ\circ}, y \leq b^{\circ\circ}, a, b \in L\}\\ &\leq \sup\{\mu(a \wedge b) : x \wedge y \leq a^{\circ\circ} \wedge b^{\circ\circ} = (a \wedge b)^{\circ\circ}, a, b \in L\}\\ &= \mu_{\circ\circ}(x \wedge y).\end{aligned}$$

Thus $\mu_{\circ\circ}$ is fuzzy ideal of L.

(2) Clearly, $(\mu \cap \theta)_{\circ\circ} \subseteq \mu_{\circ\circ} \cap \theta_{\circ\circ}$. Let $x \in L$. Then

$$\begin{aligned}&(\mu_{\circ\circ} \cap \theta_{\circ\circ})(x) = \mu_{\circ\circ}(x) \wedge \theta_{\circ\circ}(x)\\ &= \sup\{\mu(a) : x \leq a^{\circ\circ}, a \in L\} \wedge \sup\{\theta(b) : x \leq b^{\circ\circ}, b \in L\}\\ &\leq \sup\{\mu(a \wedge b) : x \leq (a \wedge b)^{\circ\circ}, a, b \in L\} \wedge \sup\{\theta(a \wedge b) : x \leq (a \wedge b)^{\circ\circ}, a, b \in L\}.\end{aligned}$$

Since $x \leq a^{\circ\circ}$ and $x \leq b^{\circ\circ}$,

$$\begin{aligned}x \leq (a \wedge b)^{\circ\circ} &= \sup\{\mu(a \wedge b) \wedge \theta(a \wedge b) : x \leq (a \wedge b)^{\circ\circ}\}\\ &= \sup\{(\mu \cap \theta)(a \wedge b) : x \leq (a \wedge b)^{\circ\circ}\}\\ &= (\mu \cap \theta)_{\circ\circ}(x).\end{aligned}$$

Thus $(\mu \cap \theta)_{\circ\circ} \subseteq \mu_{\circ\circ} \cap \theta_{\circ\circ}$.

(3) Let $x \in L$. Then

$$\begin{aligned}(\mu_{\circ\circ} \vee \theta_{\circ\circ})(x) &= \sup\{\mu_{\circ\circ}(x_1) \wedge \theta_{\circ\circ}(x_2) : x = x_1 \vee x_2\}\\ &= \sup\{\mu(a_1) : x_1 \leq a_1^{\circ\circ}\} \wedge \sup\{\theta(a_2) : x_2 \leq a_2^{\circ\circ}\} : x_1 \vee x_2 = x\}\\ &= \sup\{\sup\{\mu(a_1) \wedge \theta(a_2) : x_1 \leq a_1^{\circ\circ}, x_2 \leq a_2^{\circ\circ}\} : x = x_1 \vee x_2\}\\ &= \sup\{\mu(a_1) \wedge \theta(a_2) : x_1 \leq a_1^{\circ\circ}, x_2 \leq a_2^{\circ\circ}, where\ x = x_1 \vee x_2\}.\end{aligned}$$

Put $A = \{\alpha,\ where\ \alpha = \mu(a_1) \wedge \theta(a_2) : x_1 \leq a_1^{\circ\circ}, x_2 \leq a_2^{\circ\circ}, where\ x = x_1 \vee x_2\}$. Then

$$\begin{aligned}(\mu \vee \theta)_{\circ\circ}(x) &= \sup\{(\mu \vee \theta)(a) : x \leq a^{\circ\circ}\}\\ &= \sup\{\sup\{\mu(a_1) \wedge \theta(a_2) : a_1 \vee a_2 = a\} : x \leq a^{\circ\circ}\}\\ &= \sup\{\mu(a_1) \wedge \theta(a_2) : a_1 \vee a_2 = a, x \leq a^{\circ\circ}\}.\end{aligned}$$

Put $B = \{\lambda,\ where\ \lambda = \mu(a_1) \wedge \theta(a_2) : a_1 \vee a_2 = a, x \leq a^{\circ\circ}\}$ and we prove that $A = B$. Let $\alpha \in A$. Then $\alpha = \mu(a_1) \wedge \theta(a_2)$ such that $x_1 \leq a_1^{\circ\circ}, x_2 \leq a_2^{\circ\circ}, where\ x = x_1 \vee x_2$. Thus $x = x_1 \vee x_2 \leq a_1^{\circ\circ} \vee a_2^{\circ\circ} = (a_1 \vee a_2)^{\circ\circ} = (a)^{\circ\circ}, a = a_1 \vee a_2$. So $\alpha \in B$, i.e., $A \subseteq B$.

Let $\lambda \in B$. Then $\lambda = \mu(a_1) \wedge \theta(a_2)$ such that $a_1 \vee a_2 = a, x \leq a^{\circ\circ}$. Thus $x \leq (a_1 \wedge a_2)^{\circ\circ} = (a_1)^{\circ\circ} \vee (a_2^{\circ\circ})$. So $x = x \wedge (a_1^{\circ\circ} \vee a_2^{\circ\circ}) = (x \wedge a_1^{\circ\circ}) \vee (x \wedge a_2^{\circ\circ})$. Let $x_1 = x \wedge a_1^{\circ\circ}, x_2 = x \wedge a_2^{\circ\circ}$. Since $x = x_1 \vee x_2$, $x_1 \leq a_1^{\circ\circ}, x_2 \leq a_2^{\circ\circ},\ where\ a = a_1 \vee a_2$. This implies $\lambda \in A$ and $B \subseteq A$. Hence $A = B$., i.e., $\sup A = \sup B$. Therefore we get $(\mu \vee \theta)_{\circ\circ} = \mu_{\circ\circ} \vee \theta_{\circ\circ}$. □

Corollary 3.4. *If $\{\mu_i : i \in \Omega\}$ is a family of fuzzy ideals of L, then*

$$\{\cap_{i \in \Omega} \mu_i\}_{\circ\circ} = \cap_{i \in \Omega} (\mu_i)_{\circ\circ}.$$

Definition 3.5. A fuzzy ideal μ of an MS-algebra L is called a dominator fuzzy ideal, if $\mu = \mu_{\circ\circ}$.

Theorem 3.6. *μ is dominator fuzzy ideal in MS-algebra L if and only if $\mu_\alpha, \alpha \in Im(\mu)$, is a dominator ideal of L.*

Proof. Suppose μ is dominator fuzzy ideal of L. Then $\mu_\alpha = (\mu_{\circ\circ})_\alpha, \forall\ \alpha \in Im(\mu)$. Clearly, $\mu_\alpha \subseteq (\mu_\alpha)_{\circ\circ}$. Let $x \in (\mu_\alpha)_{\circ\circ}$. Thus there exists $y \in \mu_\alpha$ such that $x \leq y^{\circ\circ}$. So $\mu_{\circ\circ}(x) = \sup\{\mu(y) : x \leq y^{\circ\circ}\} \geq \mu(y) \geq \alpha$. Hence $x \in (\mu_{\circ\circ})_\alpha = \mu_\alpha$. Therefore $(\mu_\alpha)_{\circ\circ} = \mu_\alpha$.

Conversely, suppose $(\mu_\alpha)_{\circ\circ} = \mu_\alpha$. Then, clearly, $\mu \subseteq \mu_{\circ\circ}$. Now let $\alpha = (\mu_{\circ\circ})(x) = \sup\{\mu(y) : x \leq y^{\circ\circ}\}$. Then for each $\epsilon > 0$, there exists $a \in L$ such that $\alpha - \epsilon \leq \mu(a)\ for\ x \leq a^{\circ\circ}$. Since ϵ is arbitrary, $\alpha \leq \mu(a)\ for\ x \leq a^{\circ\circ}$. Thus $a \in \mu_\alpha$, for $x \leq a^{\circ\circ}$. So $x \in (\mu_\alpha)_{\circ\circ} = \mu_\alpha$. Hence $\mu(x) \geq \alpha = (\mu_{\circ\circ})(x)$. Therefore $\mu = \mu_{\circ\circ}$. □

Corollary 3.7. *I is dominator ideal of MS-algebra L if and only if χ_I is dominator fuzzy ideal of L.*

We denote the set of all dominator fuzzy ideals of L by $\mu_{\circ\circ}(L)$. Now we have the following theorem.

Theorem 3.8. *The set of all dominator fuzzy ideals of L, $\mu_{\circ\circ}(L)$, is a bounded distributive lattice with set inclusion.*

Proof. It is obvious by Lemma 3.3. □

4. Closure fuzzy ideals of MS-algebra

In this section, the notion of closure fuzzy ideals is introduced in MS-algebras and some properties are observed. We denote the set of all fuzzy ideals of L by $FI(L)$ and the set of all fuzzy ideals of $M_{\circ\circ}(L)$ by $FI(M_{\circ\circ})(L))$.

Definition 4.1. Let L be an MS-algebra. For any fuzzy ideal μ of L, define an operator $\sigma : FI(L) \to FI(M_{\circ\circ}(L))$ as follows: for all $a \in L$,

$$\sigma(\mu)((a]_{\circ\circ}) = \sup\{\mu(b) : (a]_{\circ\circ} = (b]_{\circ\circ},\ b \in L\}.$$

Definition 4.2. Let L be an MS-algebra. For any fuzzy ideal θ of $M_{\circ\circ}(L)$, define an operator $\overleftarrow{\sigma} : FI(M_{\circ\circ}(L)) \to FI(L)$ as follows: $\overleftarrow{\sigma}(\theta)(a) = \theta((a]_{\circ\circ})$, for all $a \in L$.

Lemma 4.3. *Let L be an MS-algebra. Then we have the following:*

(1) *for any fuzzy ideal μ of L, $\sigma(\mu)$ is fuzzy ideal of $M_{\circ\circ}(L)$,*
(2) *for any fuzzy ideal θ of $M_{\circ\circ}(L)$, $\overleftarrow{\sigma}(\theta)$ is fuzzy ideal of L,*
(3) *$\overleftarrow{\sigma}$ and σ are isotones,*
(4) *$\sigma(\overleftarrow{\sigma})(\mu) = \mu$, for all fuzzy ideal μ of $M_{\circ\circ}(L)$,*
(5) *σ is a homomorphism from fuzzy ideal of L into fuzzy ideal of $M_{\circ\circ}(L)$.*

Proof. Let $a, b \in L$. Then $(a]_{\circ\circ}, (b]_{\circ\circ} \in M_{\circ\circ}(L)$.

(1) Clearly, $\sigma(\mu)((0]_{\circ\circ}) = 1$. Then

$$\begin{aligned}\sigma(\mu)((a]_{\circ\circ}) \wedge \sigma(\mu)((b]_{\circ\circ}) &= \sup\{\mu(x) : (x]_{\circ\circ} = (a]_{\circ\circ}\} \wedge \sup\{\mu(y) : (y]_{\circ\circ} = (a]_{\circ\circ}\} \\ &= \sup\{\mu(x) \wedge \mu(y) : (x]_{\circ\circ} = (a]_{\circ\circ}, (y]_{\circ\circ} = (a]_{\circ\circ}\} \\ &\leq \sup\{\mu(x \vee y) : (x]_{\circ\circ} = (a]_{\circ\circ}, (y]_{\circ\circ} = (a]_{\circ\circ}\} \\ &\leq \sup\{\mu(x \vee y) : (x \vee y]_{\circ\circ} = (a \vee b]_{\circ\circ}\} \\ &= \sigma(\mu)((a \vee b]_{\circ\circ})\end{aligned}$$

$$= \sigma(\mu)((a]_{\circ\circ} \vee (b]_{\circ\circ})$$

and

$$\begin{aligned}
&\sigma(\mu)((a]_{\circ\circ}) \vee \sigma(\mu)((b]_{\circ\circ})\\
&= \sup\{\mu(x) : (x]_{\circ\circ} = (a]_{\circ\circ}\} \vee \sup\{\mu(y) : (y]_{\circ\circ} = (b]_{\circ\circ}\}\\
&\leq \sup\{\mu(x \wedge y) : (x \wedge y]_{\circ\circ} = (a \wedge b]_{\circ\circ}\} \vee \sup\{\mu(x \wedge y) : (x \wedge y]_{\circ\circ} = (a \wedge b]_{\circ\circ}\}\\
&= \sup\{\mu(x \wedge y) : (x \wedge y]_{\circ\circ} = (a \wedge b]_{\circ\circ}\}\\
&= \sigma(\mu)((a \wedge b]_{\circ\circ})\\
&= \sigma(\mu)((a]_{\circ\circ} \wedge (b]_{\circ\circ}).
\end{aligned}$$

Thus $\sigma(\mu)$ is fuzzy ideal of $M_{\circ\circ}(L)$.

(2) Clearly, $\overleftarrow{\sigma}(\theta)(0)) = \theta((0]_{\circ\circ}) = 1$

and

$$\overleftarrow{\sigma}(\theta)(a \vee b) = \theta((a \vee b]_{\circ\circ}) = \theta((a]_{\circ\circ}) \wedge \theta((b]_{\circ\circ}) = \overleftarrow{\sigma}(\theta)(a) \wedge \overleftarrow{\sigma}(\theta)(b).$$

(3) Let μ, θ be fuzzy ideals of $M_{\circ\circ}(L)$ such that $\mu \subseteq \theta$. Then

$$\overleftarrow{\sigma}(\mu)(a) = \mu((a]_{\circ\circ}) \subseteq \theta((a]_{\circ\circ}) = \overleftarrow{\sigma}(\theta)(a).$$

Similarly, σ is an isotone.

(4) Let $(a]_{\circ\circ} \in M_{\circ\circ}(L)$ and let μ fuzzy ideal of $M_{\circ\circ}(L)$. Then

$$\begin{aligned}
\sigma\overleftarrow{\sigma}(\mu)((a]_{\circ\circ}) &= \sup\{\overleftarrow{\sigma}(\mu)(b) : (a]_{\circ\circ} = (a]_{\circ\circ}\}\\
&= \sup\{\sigma(\mu)(b]_{\circ\circ}) : (a]_{\circ\circ} = (b]_{\circ\circ}\}\\
&= \mu((a]_{\circ\circ}).
\end{aligned}$$

Thus $\sigma(\overleftarrow{\sigma})(\mu) = \mu$, for all fuzzy ideals μ of $M_{\circ\circ}(L)$.

(5) Let $\mu, \theta \in FI(L)$. Since σ is isotone, $\sigma(\mu \cap \theta) \subseteq \sigma(\mu) \cap \sigma(\theta)$. On the other hand,

$$\begin{aligned}
&(\sigma(\mu) \cap \sigma(\theta))((a]_{\circ\circ})\\
&= \sigma(\mu)((a]_{\circ\circ}) \wedge \sigma(\theta)((a]_{\circ\circ})\\
&= \sup\{\mu(x) : (x]_{\circ\circ} = (a]_{\circ\circ}\} \wedge \sup\{\theta(y) : (y]_{\circ\circ} = (a]_{\circ\circ}\}\\
&\leq \sup\{\mu(x \wedge y) : (x \wedge y]_{\circ\circ} = (a]_{\circ\circ}\} \wedge \sup\{\theta(x \wedge y) : (x \wedge y]_{\circ\circ} = (a]_{\circ\circ}\}\\
&= \sup\{\mu(x \wedge y) \wedge \theta(x \wedge y) : (x \wedge y]_{\circ\circ} = (a]_{\circ\circ}\}\\
&= \sup\{(\mu \cap \theta)(x \wedge y) : (x \wedge y]_{\circ\circ} = (a]_{\circ\circ}\}\\
&= \sigma(\mu \cap \theta)((a]_{\circ\circ}).
\end{aligned}$$

Then $\sigma(\mu) \cap \sigma(\theta) \subseteq \sigma(\mu \cap \theta)$. Also clearly, $\sigma(\mu) \vee \sigma(\theta) \subseteq \sigma(\mu \vee \theta)$. Again

$$\begin{aligned}
&\sigma(\mu \vee \theta)((x]_{\circ\circ})\\
&= \sup\{(\mu \vee \theta)(y) : (y]_{\circ\circ} = (x]_{\circ\circ}\}\\
&= \sup\{\sup\{\mu(a) \wedge \theta(b) : y = a \vee b\} : (y]_{\circ\circ} = (x]_{\circ\circ}\}\\
&\leq \sup\{\sup\{\mu(a_1) \wedge \theta(b_1) : (a_1]_{\circ\circ} = (a]_{\circ\circ}, (b_1]_{\circ\circ} = (b]_{\circ\circ}\} : (y]_{\circ\circ} = (x]_{\circ\circ}\}\\
&= \sup\{\sup\{\mu(a_1) : (a_1]_{\circ\circ} = (a]_{\circ\circ}\} \wedge \sup\{\theta(b_1) : (b_1]_{\circ\circ} = (b]_{\circ\circ}\} :\\
&\qquad\qquad (a]_{\circ\circ} \vee (b]_{\circ\circ} = (x]_{\circ\circ}\}\\
&= \sup\{\sigma(\mu)((a]_{\circ\circ}) \wedge \sigma(\theta)((b]_{\circ\circ}) : (a]_{\circ\circ} \vee (b]_{\circ\circ} = (x]_{\circ\circ}\}\\
&= (\sigma(\mu) \vee \sigma(\theta))((x]_{\circ\circ}).
\end{aligned}$$

Thus $\sigma(\mu \vee \theta) \subseteq \sigma(\mu) \vee \sigma(\theta)$. So $\sigma(\mu \vee \theta) = \sigma(\mu) \vee \sigma(\theta)$. Hence σ is a homomorphism from the lattice of fuzzy ideals of L into the lattice of fuzzy ideals of $M_{\circ\circ}(L)$. □

Theorem 4.4. *The map $\overleftarrow{\sigma}\sigma : FI(L) \to FI(L)$ is a closure operator, that is, for any $\mu, \theta \in FI(L)$,*

(1) $\mu \subseteq \overleftarrow{\sigma}\sigma(\mu)$,
(2) $\mu \subseteq \theta$ *implies* $\overleftarrow{\sigma}\sigma(\mu) \subseteq \overleftarrow{\sigma}\sigma(\theta)$,
(3) $\overleftarrow{\sigma}\sigma\{\overleftarrow{\sigma}\sigma(\mu)\} = \overleftarrow{\sigma}\sigma(\mu)$.

Proof. The prove of (1) and (2) is straight forward.

(3) For any $x \in L$,

$$\begin{aligned} \overleftarrow{\sigma}\sigma\{\overleftarrow{\sigma}\sigma(\mu)\}(x) &= \sigma\{\overleftarrow{\sigma}\sigma(\mu)\}(((x]_{\circ\circ}) \\ &= \sup\{\overleftarrow{\sigma}\sigma(\mu)(y) : (x]_{\circ\circ}) = (y]_{\circ\circ}\} \\ &= \sup\{\sigma(\mu)(((y]_{\circ\circ}) : ((x]_{\circ\circ} = ((y]_{\circ\circ}\} \\ &= \sigma(\mu)((x]_{\circ\circ}) \\ &= \overleftarrow{\sigma}\sigma(\mu)(x). \end{aligned}$$

□

Corollary 4.5. *For any two fuzzy ideals μ and θ of an MS-algebra L, we have*

$$\overleftarrow{\sigma}\sigma(\mu \cap \theta) = \overleftarrow{\sigma}\sigma(\mu) \cap \overleftarrow{\sigma}\sigma(\theta).$$

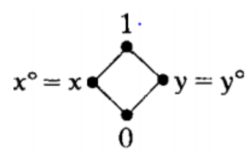
Proof. By Lemma 4.3 (5), $\sigma(\mu \cap \theta) = \sigma(\mu) \cap \sigma(\theta)$. Now for any $y \in L$,

$$\begin{aligned} \overleftarrow{\sigma}\sigma(\mu \cap \theta)(y) &= \sigma(\mu \cap \theta)((y]_{\circ\circ}) \\ &= \sigma(\mu)((y]_{\circ\circ}) \wedge \sigma(\theta)((y]_{\circ\circ}) \\ &= \overleftarrow{\sigma}\sigma(\mu)(y) \wedge \overleftarrow{\sigma}\sigma(\theta)(y). \end{aligned}$$

Then $\overleftarrow{\sigma}\sigma(\mu \cap \theta) = \overleftarrow{\beta}\sigma(\mu) \cap \overleftarrow{\sigma}\sigma(\theta)$. □

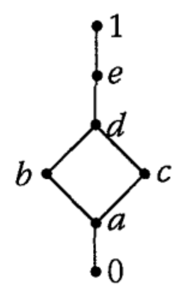
Definition 4.6. A fuzzy ideal μ of L is called a closure fuzzy ideal, if $\overleftarrow{\sigma}\sigma(\mu) = \mu$.

Example 4.7. Let us consider the following figure



Then clearly $(L, \vee, \wedge, ^\circ, 0, 1)$ is an MS-algebra. Define a fuzzy subset μ of L as $\mu(x) = \mu(0) = 1$ and $\mu(1) = \mu(y) = 0.5$. We can easily verify that μ is closure fuzzy ideal of L.

Example 4.8. Let us consider the following figure with given table:



x	0	a	b	c	d	e	1
x°	1	e	e	e	e	e	0

Then clearly, $(L, \vee, \wedge, ^\circ, 0, 1)$ is an MS-algebra. Let $\mu : L \to [0, 1]$ defined as $\mu(a) = \mu(b) = 0.8, \mu(c) = \mu(d) = 0.6,\ \mu(e) = \mu(1) = 0$ and $\mu(0) = 1$. We can easily verify that μ is fuzzy ideal of L but not closure fuzzy ideal of L.

In the following Theorems we characterize closure fuzzy ideal in terms of level subsets and characteristic functions.

Theorem 4.9. *For proper fuzzy subsets μ of L, μ is closure fuzzy ideal if and only if $\mu_\alpha, \forall\alpha \in Im(\mu)$, is closure ideal of L.*

Proof. Suppose μ is closure fuzzy ideal of L. Then $(\overleftarrow{\sigma}\sigma(\mu))_\alpha = \mu_\alpha$. To prove each level subsets of μ is closure fuzzy ideal of L, it is enough to show $\overleftarrow{\sigma}\sigma(\mu_\alpha) = \mu_\alpha$. Clearly, $\mu_\alpha \subseteq \overleftarrow{\sigma}\sigma(\mu_\alpha)$. Let $x \in \overleftarrow{\sigma}\sigma(\mu_\alpha)$. Then $(x]_{\circ\circ} \in \sigma(\mu_\alpha)$. Thus there exists $y \in \mu_\alpha$ such that $(x]_{\circ\circ} = (y]_{\circ\circ}$, i.e., $\mu(y) \geq \alpha$ such that $(x]_{\circ\circ} = (y]_{\circ\circ}$. So $\sigma(\mu)((x]_{\circ\circ}) = \sup\{\mu(x) : (x]_{\circ\circ} = (y]_{\circ\circ}\} \geq \alpha$ and thus $\overleftarrow{\sigma}\sigma(\mu)(x) \geq \alpha$. Hence $x \in (\overleftarrow{\sigma}\sigma(\mu))_\alpha = \mu_\alpha$, i.e., $\overleftarrow{\sigma}\sigma(\mu_\alpha) \subseteq \mu_\alpha$. Therefore $\overleftarrow{\sigma}\sigma(\mu_\alpha) = \mu_\alpha$.

Conversely, clearly $\mu \subseteq \overleftarrow{\sigma}\sigma(\mu)$. Let $\alpha = \overleftarrow{\sigma}\sigma(\mu)(x) = \sup\{\mu(y) : (y]_{\circ\circ} = (x]_{\circ\circ}\}$. Then for each $\epsilon > 0$, there is $a \in L, (a]_{\circ\circ} = (x]_{\circ\circ}$ such that $\mu(a) \geq \alpha - \epsilon$. Since ϵ is arbitrary, $\mu(a) \geq \alpha$ such that $(a]_{\circ\circ} = (x]_{\circ\circ}$. Thus $a \in \mu_\alpha$. So $x \in \overleftarrow{\sigma}\sigma(\mu_\alpha) = \mu_\alpha$. Hence $\mu(x) \geq \alpha = \overleftarrow{\sigma}\sigma(\mu)(x)$. Therefore $\mu = \overleftarrow{\sigma}\sigma(\mu)$. □

Theorem 4.10. *For a nonempty subset I of L, I is a closure ideal if and only if χ_I is closure fuzzy ideal of L.*

Theorem 4.11. *Let μ be fuzzy ideal of an MS-algebra L. Then the following conditions are equivalent:*

(1) *μ is a closure fuzzy ideal,*
(2) *For all $x, y \in L, (x]_{\circ\circ} = (y]_{\circ\circ}$ implies $\mu(x) = \mu(y)$,*
(3) *$\mu = \mu_{\circ\circ}$.*

Proof. (1) $\Rightarrow$ (2): Suppose μ is a closure fuzzy ideal. Let $x, y \in L$ such that $(x]_{\circ\circ} = (y]_{\circ\circ}$. Then

$$\mu(x) = \overleftarrow{\sigma}\sigma(\mu)(x) = \sigma(\mu)((x]_{\circ\circ}) = \sigma(\mu)((y]_{\circ\circ}) = \overleftarrow{\sigma}\sigma(\mu)(y) = \mu(y).$$

(2) $\Rightarrow$ (1): Suppose for all $x, y \in L, (x]_{\circ\circ} = (y]_{\circ\circ}$ implies $\mu(x) = \mu(y)$. Then

$$\overleftarrow{\sigma}\sigma(\mu)(x) = \sigma(\mu)((x]_{\circ\circ}) = \sup\{\mu(y) : (x]_{\circ\circ} = (y]_{\circ\circ}\} = \mu(x).$$

(1) $\Rightarrow$ (3): Suppose μ is closure fuzzy ideal of L. By Lemma 3.2, $\mu \subseteq \mu_{\circ\circ}$. Let $\forall x \in L$. Then

$$\begin{aligned}\mu_{\circ\circ}(x) &= \sup\{\mu(y) : x \leq y^{\circ\circ}\}\\ &= \sup\{\mu(y) : (x]_{\circ\circ} \subseteq (y]_{\circ\circ}\}\\ &\leq \sigma(\mu)((x]_{\circ\circ}) = \overleftarrow{\sigma}\sigma(\mu)(x)\\ &= \mu(x).\end{aligned}$$

Thus $\mu_{\circ\circ} \subseteq \mu$. So $\mu_{\circ\circ} = \mu$.

(3) $\Rightarrow$ (1): Suppose $\mu = \mu_{\circ\circ}$, for all $x \in L$. Then

$$\begin{aligned}\overleftarrow{\sigma}\sigma(\mu)(x) &= \sigma(\mu)((x]_{\circ\circ})\\ &= \sup\{\mu(y) : (x]_{\circ\circ} = (y]_{\circ\circ}\}\\ &\leq \sup\{\mu(y) : x \leq y^{\circ\circ}\}\\ &= \mu_{\circ\circ}(x) = \mu(x).\end{aligned}$$

It is clear that $\mu \subseteq \overleftarrow{\sigma}\sigma(\mu)$. Thus $\mu = \overleftarrow{\sigma}\sigma(\mu)$. By transitivity, (2) $\Rightarrow$ (3) $\Rightarrow$ (2). □

Lemma 4.12. *The following conditions are held in any MS-algebra L:*

(1) *for any fuzzy ideal μ of L, $\overleftarrow{\sigma}\sigma(\mu) = \mu_{\circ\circ}$,*
(2) *for any fuzzy ideal μ of L, $\mu_{\circ\circ}$ is a closure fuzzy ideal,*
(3) *the map $\overleftarrow{\sigma}\sigma : FI(L) \to FI(L)$ is a (0,1)-homomorphism.*

Proof. (1) $\overleftarrow{\sigma}\sigma(\mu)(x) = \sup\{\mu(y) : (y]_{\circ\circ} = (x]_{\circ\circ}\} \leq \sup\{\mu(y) : x \leq y^{\circ\circ}\} = \mu_{\circ\circ}(x)$, for all $x \in L$. On the other hand, for all $x \in L$,

$$\begin{aligned}\mu_{\circ\circ}(x) &= \sup\{\mu(y) : x \leq y^{\circ\circ}\}\\ &= \sup\{\mu(y) : (x]_{\circ\circ} \subseteq (y^{\circ\circ}] = (y]_{\circ\circ}\}\\ &= \sigma((x]_{\circ\circ})\\ &= \overleftarrow{\sigma}\sigma(\mu)(x).\end{aligned}$$

Then $\overleftarrow{\sigma}\sigma(\mu) = \mu_{\circ\circ}$.

(2) From (1) $\overleftarrow{\sigma}\sigma(\mu_{\circ\circ}) = \{\mu_{\circ\circ}\}_{\circ\circ}$. By Lemma 3.2 (3), $\{\mu_{\circ\circ}\}_{\circ\circ} = \mu_{\circ\circ}$. Then $\overleftarrow{\sigma}\sigma(\mu_{\circ\circ}) = \mu_{\circ\circ}$. Thus $\mu_{\circ\circ}$ is a closure fuzzy ideal of L.

(3) Clearly, $\overleftarrow{\sigma}\sigma(\chi_{\{0\}}) = \chi_{\{0\}}$ and $\overleftarrow{\sigma}\sigma(\chi_{\{L\}}) = \chi_{\{L\}}$. Let $\mu, \theta \in FI(L)$. Then by (2) and Lemma 3.3 (2),(3), we get

$$\overleftarrow{\sigma}\sigma(\mu \cap \theta) = (\mu \cap \theta)_{\circ\circ} = \mu_{\circ\circ} \cap \theta_{\circ\circ} = \overleftarrow{\sigma}\sigma(\mu) \cap \overleftarrow{\sigma}\sigma(\theta)$$

and

$$\overleftarrow{\sigma}\sigma(\mu \vee \theta) = (\mu \vee \theta)_{\circ\circ} = \mu_{\circ\circ} \vee \theta_{\circ\circ} = \overleftarrow{\sigma}\sigma(\mu) \vee \overleftarrow{\sigma}\sigma(\theta).$$

Thus $\overleftarrow{\sigma}\sigma$ is a (0,1)-homomorphism. □

Now, we denote the set of all closure fuzzy ideals of L by $FI_C(L)$. Then we have the following theorem

Theorem 4.13. *Let L be an MS-algebra, the set $FI_C(L)$ forms a bounded distributive lattice.*

Definition 4.14. A proper closure fuzzy ideal μ of L is called prime closure fuzzy ideal, if for any two fuzzy ideals η, ν of L, $\eta \cap \nu \subseteq \mu$ implies $\eta \subseteq \mu$ or $\nu \subseteq \mu$.

Example 4.15. In Example 4.8, clearly $A = \{0\}$, $B = \{0, a\}$, $C = \{0, a, b\}$, $D = \{0, a, c\}$, $E = \{0, a, b, c, d\}$, $F = \{0, a, b, c, d, e\}$ are ideals of L and A, E and F are prime ideals of L. By the Theorem 2.16, A^1_α, E^1_α and F^1_α are prime fuzzy ideals of L, for any $\alpha \in [0, 1)$. Now, we prove that A^1_α, E^1_α and F^1_α are prime closure fuzzy ideals or not. We have:

$$\overleftarrow{\sigma}\sigma(A^1_\alpha)(0) = \sup\{A^1_\alpha(x) : (x]_{\circ\circ} = (0]_{\circ\circ}, x \in L\} = 1 = A^1_\alpha(0)$$

and for any $x \in L, x \neq 0$,

$$\overleftarrow{\sigma}\sigma(A^1_\alpha)(x) = \sup\{A^1_\alpha(y) : (x]_{\circ\circ} = (y]_{\circ\circ}, y \in L\} = \alpha = A^1_\alpha(x).$$

Then $\overleftarrow{\sigma}\sigma(A^1_\alpha) = A^1_\alpha$. Since $(e]_{\circ\circ} = (a]_{\circ\circ} = (b]_{\circ\circ} = (c]_{\circ\circ} = (d]_{\circ\circ}$ and $e \notin E$,

$$\begin{aligned}\overleftarrow{\sigma}\sigma(E^1_\alpha)(e) &= \sup\{E^1_\alpha(y) : (e]_{\circ\circ} = (y]_{\circ\circ}, y \in L\}\\ &= \sup\{E^1_\alpha(a), E^1_\alpha(b), E^1_\alpha(c), E^1_\alpha(d), E^1_\alpha(e)\}\\ &= \{1, 1, 1, 1, \alpha\}\\ &= 1\\ &\neq \alpha\\ &= E^1_\alpha(e).\end{aligned}$$

Thus $\overleftarrow{\sigma}\sigma(E^1_\alpha) \neq E^1_\alpha$. Similarly, $\overleftarrow{\sigma}\sigma(F^1_\alpha) = F^1_\alpha$. So A^1_α and F^1_α are prime closure fuzzy ideals. Hence E^1_α is prime fuzzy ideal of L but not prime closure fuzzy ideal of L.

Theorem 4.16. *Let L be an MS-algebra. Then there is an isomorphism of the lattice of closure fuzzy ideals of L onto the lattice of fuzzy ideals on $M_{\circ\circ}(L)$. Under* *this isomorphism the prime closure fuzzy ideals of L corresponding to prime fuzzy ideals of $M_{\circ\circ}(L)$.*

Proof. Define the mapping $h : FI_C(L) \to FI(M_{\circ\circ}(L))$ by $h(\mu) = \sigma(\mu), \forall \mu \in FI_C(L)$. Obviously, $h(\chi_{\{0\}}) = \chi_{\{0\}}$ and $h(\chi_L) = \sigma(\chi_L) = \chi_{M_{\circ\circ}(L)}$. Let μ and θ be any two closure fuzzy ideals of L. Since σ is a homomrphism, we have get

$$h(\mu \cap \theta) = \sigma(\mu \cap \theta) = \sigma(\mu) \cap \sigma(\theta) = h(\mu) \cap h(\theta)$$

and

$$h(\mu \vee \theta) = \sigma(\mu \vee \theta) = \sigma(\mu) \vee \sigma(\theta) = h(\mu) \vee h(\theta).$$

Then h is a $(0,1)$-lattice homomorphism. Let $\mu, \theta \in FI_C(L)$ such that $h(\mu) = h(\theta)$. Then $\sigma(\mu) = \sigma(\theta)$. Thus $\mu = \overleftarrow{\sigma}\sigma(\mu) = \overleftarrow{\sigma}\sigma(\theta) = \theta$. So h is one to one.

Now we prove that h is an onto. Let $\mu \in FI(M_{\circ\circ}(L))$. Then $\overleftarrow{\sigma}(\mu)$ is fuzzy ideal of L, by Lemma 4.3 (1) and by Lemma 4.3 (3), $\sigma\overleftarrow{\sigma}(\mu) = \mu$. Thus $\overleftarrow{\sigma}\{\sigma\overleftarrow{\sigma}(\mu)\} = \overleftarrow{\sigma}(\mu)$. So $\overleftarrow{\sigma}(\mu) \in FI_C(L)$. Hence h is onto. Therefore h is isomorphism.

Also we have obtained one-to-one correspondence between prime closure fuzzy ideals of L and the prime fuzzy ideals of $M_{\circ\circ}(L)$. Let μ be any prime closure fuzzy ideal of L. We need to prove that $\sigma(\mu)$ is prime ideal of $M_{\circ\circ}(L)$. Let $\theta,\ \nu$ are any ideals of $M_{\circ\circ}(L)$ such that $\theta \cap \nu \subseteq \sigma(\mu)$. Since h is onto, there exist a closure fuzzy ideals of L , θ_1, ν_1 such that $\sigma(\theta_1) = \theta$ and $\sigma(\nu_1) = \nu$. Now $\sigma(\theta_1) \cap \sigma(\nu_1) \subseteq \sigma(\mu)$. Then $\sigma(\theta_1 \cap \nu_1) \subseteq \sigma(\mu)$. Thus $\theta_1 \cap \nu_1 \subseteq \mu$. So $\theta_1 \subseteq \mu$ or $\nu_1 \subseteq \mu$. Hence $\sigma(\theta_1) \subseteq \sigma(\mu)$ or $\sigma(\nu_1) \subseteq \sigma(\mu)$. Therefore $\sigma(\mu)$ is prime fuzzy ideal of L.

Conversely, let μ be any prime ideal of $M_{\circ\circ}(L)$. Then there exists any closure fuzzy ideal μ_1 of L such that $\sigma(\mu_1) = \mu$. Let $\phi,\ \xi$ any fuzzy ideals of L such that $\phi \cap \xi \subseteq \mu_1$. Then $\sigma(\phi \cap \xi) = \sigma(\phi) \cap \sigma(\xi) \subseteq \sigma(\mu_1)$. Since $\sigma(\mu_1)$ is prime ideal of L, $\sigma(\phi) \subseteq \sigma(\mu_1)$ or $\sigma(\xi) \subseteq \sigma(\mu_1)$. Thus $\phi \subseteq \overleftarrow{\sigma}\sigma(\phi) \subseteq \overleftarrow{\sigma}\sigma(\mu_1) = \mu_1$ or $\xi \subseteq \overleftarrow{\sigma}\sigma(\xi) \subseteq \overleftarrow{\sigma}\sigma(\mu_1) = \mu_1$. So μ_1 is prime closure fuzzy ideal of L. Hence prime closure fuzzy ideals of L one to one correspondence to prime ideals of $M_{\circ\circ}(L)$. □

5. Closure fuzzy ideals and homomorphisms of MS-algebras

In this section, some properties of the homomorphic images and the inverse images of closure fuzzy ideals are observed. Homomorphism on an MS-algebra L, that means a lattice homomorphism f which preserves "$\circ$" such that $(f(x))^\circ = f(x^\circ)$, for all $x \in L$.

Lemma 5.1. *Let L and M be two MS-algebras and $f : L \to M$ be a homomorphism. Then we have the following:*

1. *for any non empty fuzzy subset μ of L, $f(\mu_{\circ\circ}) \subseteq \{f(\mu)\}_{\circ\circ}$,*
2. *for any non empty fuzzy subset θ of M, $\{f^{-1}(\theta)\}_{\circ\circ} \subseteq f^{-1}(\theta_{\circ\circ})$.*

Proof. (1) Let $y \in M$.

Case 1: Suppose $f^{-1}(y) = \varnothing$. Then $f(\mu_{\circ\circ})(y) = 0 \leq \{f(\mu)\}_{\circ\circ}(y)$.

Case 2: Suppose $f^{-1}(y) \neq \varnothing$. Then

$$\begin{aligned} f(\mu_{\circ\circ})(y) &= \sup\{\mu_{\circ\circ}(b) : f(b) = y, b \in L\} \\ &= \sup\{\sup\{\mu(a) : b \leq a^{\circ\circ}, a \in L\} : f(b) = y, b \in L\} \\ &= \sup\{\mu(a) : b \leq a^{\circ\circ}, f(b) = y, a, b \in L\} \end{aligned}$$

and

$$\begin{aligned}\{f(\mu)\}_{\circ\circ}(y) &= \sup\{f(\mu)(x) : y \leq x^{\circ\circ}, x \in M\}\\ &= \sup\{\sup\{\mu(a) : f(a) = x, a \in L\} : y \leq x^{\circ\circ}, x \in M\}\\ &= \sup\{\mu(a) : f(a) = x, y \leq x^{\circ\circ}, a \in L, x \in M\}.\end{aligned}$$

Let $A = \{\alpha, where\ \mu(a) = \alpha, b \leq a^{\circ\circ}, f(b) = y, a, b \in L\}$ and let $B = \{\lambda,\ where\ \lambda = \mu(a) : f(a) = x, y \leq x^{\circ\circ}, a \in L, x, y \in M\}$. We prove that $A \subseteq B$. Let $\alpha \in A$. Then $b \leq a^{\circ\circ}, f(b) = y, a, b \in L$. Thus $f(b) \leq f(a^{\circ\circ}) = (f(a))^{\circ\circ}, f(b) = y, a, b \in L$. So $y \leq (x)^{\circ\circ}, f(a) = x, a \in L, x \in M$. Hence $\alpha \in B$ and thus $A \subseteq B$. This implies $\sup A \leq \sup B$. Therefore $f(\mu_{\circ\circ}) \subseteq \{f(\mu)\}_{\circ\circ}$.

(2) Let $x \in L$. Then

$$\begin{aligned}\{f^{-1}(\theta)\}_{\circ\circ}(x) &= \sup\{f^{-1}(\theta)(a) : x \leq a^{\circ\circ}\}\\ &= \sup\{\theta(f(a)) : x \leq a^{\circ\circ}\}\\ &\leq \sup\{\theta(f(a)) : f(x) \leq (f(a))^{\circ\circ}\}\\ &= \theta_{\circ\circ}(f(x))\\ &= f^{-1}(\theta_{\circ\circ})(x).\end{aligned}$$

Thus $\{f^{-1}(\theta)\}_{\circ\circ} \subseteq f^{-1}(\theta_{\circ\circ})$. □

Definition 5.2. Let $f : L \to M$ be a homomorphism of an MS-algebra L into an MS-algebra M. Then f is called dominator fuzzy ideal preserving, if $f(\mu_{\circ\circ}) = \{f(\mu)\}_{\circ\circ}$.

Theorem 5.3. *Let $f : L \to M$ be a homomorphism of an MS-algebra L onto an MS-algebra M. Then f is a dominator fuzzy ideal preserving.*

Proof. Let $x, y \in M$. Then

$$\begin{aligned}\{f(\mu)\}_{\circ\circ}(y) &= \sup\{f(\mu)(x) : y \leq x^{\circ\circ}\}\\ &= \sup\{\sup\{\mu(a) : f(a) = x, a \in L\} : y \leq x^{\circ\circ}\}\\ &= \sup\{\mu(a) : y \leq f(a^{\circ\circ})\}\\ &= f(\mu_{\circ\circ}).\end{aligned}$$

Thus f is dominator fuzzy ideal preserving. □

Theorem 5.4. *Let $f : L \to M$ be a homomorphism of an MS-algebra L onto an MS-algebra M. Then for any closure fuzzy ideal μ of L, $f(\mu)$ is closure fuzzy ideal of M.*

Proof. By Lemma 4.12 (2), Theorem 5.3 and Theorem 4.11 (3),

$$\overleftarrow{\sigma}\sigma(f(\mu)) = (f(\mu))_{\circ\circ} = f(\mu_{\circ\circ}) = f(\mu).$$

□

Theorem 5.5. *Let L and M be two MS-algebras and $f : L \to M$ be a homomorphism. Then we have the following:*

(1) *for any closure fuzzy ideal θ of M, $f^{-1}(\theta)$ is a closure fuzzy ideal of L,*
(2) *for any closure fuzzy ideal θ of M, $f^{-1}(\theta_{\circ\circ}) = \{f^{-1}(\theta)\}_{\circ\circ}$.*

Proof. (1) Clearly, $f^{-1}(\theta) \subseteq \overleftarrow{\sigma}\sigma(f^{-1}(\theta))$. By Lemma 4.12 (2), Lemma 5.1 (2) and Theorem 4.11 (3), $\overleftarrow{\sigma}\sigma(f^{-1}(\theta)) = \{f^{-1}(\theta)\}_{\circ\circ} \subseteq f^{-1}(\theta_{\circ\circ}) = f^{-1}(\theta)$. Then we have $\overleftarrow{\sigma}\sigma(f^{-1}(\theta)) = f^{-1}(\theta)$.

(2) Since θ is closure fuzzy ideal of M . Then by (1), $f^{-1}(\theta)$ is closure fuzzy ideal of L. By Theorem 4.11 (3), $\{f^{-1}(\theta)\}_{\circ\circ} = f^{-1}(\theta) = f^{-1}(\theta_{\circ\circ})$. □

Theorem 5.6. *Let $f : L \to M$ be an onto homomorphism between MS-algebras L and M. Then $FI_C(L)$ is homomorphic of $FI_C(M)$.*

Proof. Define the map $h : FI_C(L) \to FI_C(M)$ by $h(\mu) = f(\mu)$, for all $\mu \in FI_C(L)$. It is clear that $h(\chi_{\{0_L\}} = \chi_{\{0_L\}}$ and $h(\chi_L) = \chi_M$. Let $\mu, \theta \in FI_C(L)$. Then for all $y \in M$,

$$\begin{aligned} h(\mu \cap \theta)(y) = f(\mu \cap \theta)(y) &= \sup\{(\mu \cap \theta)(x) : f(x) = y\} \\ &= \sup\{\mu(x) \wedge \theta(x) : f(x) = y\} \\ &= \sup\{\mu(x) : f(x) = y\} \wedge \sup\{\theta(x) : f(x) = y\} \\ &= f(\mu)(y) \wedge f(\theta)(y) = h(\mu)(y) \wedge h(\theta)(y). \end{aligned}$$

Thus $h(\mu \cap \theta) = h(\mu) \cap h(\theta)$ and for any $y \in M$,

$$\begin{aligned} (h(\mu) \vee h(\theta))(y) &= (f(\mu) \vee f(\theta))(y) \\ &= \sup\{f(\mu)(a) \wedge f(\theta)(b) : a \vee b = y, a, b \in M\} \\ &= \sup\{\sup\{\mu(a_1) : f(a_1) = a\} \wedge \sup\{\theta(b_1) : f(b_1) = b\}, \\ &\qquad a \vee b = y, a_1, b_1 \in L, a, b \in M\} \\ &= \sup\{\mu(a_1) \wedge \theta(b_1) : f(a_1) \vee f(b_1) = f(a_1 \vee b_1) = y\} \\ &= \sup\{(\mu \vee \theta)(x) : x = a_1 \vee b_1 : f(x) = y\} \\ &= f(\mu \vee \theta)(y) = h(\mu \vee \theta)(y). \end{aligned}$$

So h is a homomorphism. □

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